Product Cordial Labeling of Some Graphs

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ARTICLE INFO	**ABSTRACT**
All the graphs considered in this article are simple and undirected. Let $G = (V(G), E(G))$ be a simple undirected Graph. A function $f: V(G) \to \{0,1\}$ is called Product Cordial if the induced function $f^*: E(G) \to \{0,1\}$ defined by $f^*(uv) = f(u)f(v)$ satisfies the condition $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for any $i, j \in \{0,1\}$, where $v_f(x)$ and $e_f(x)$ denotes the number of vertices and number of edges with label x respectively. A Graph G is called Product Cordial graph if its admit Product Cordial labeling. In this article, we have discussed Product Cordial labeling of some interesting graphs.

Keywords: Product Cordial, cycle, complete bipartite graph, join of two graphs.

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1 Introduction

We begin with simple, finite, connected and undirected graph $G = (V(G), E(G))$. The concept of cordial labeling was introduced by Cahit in the year 1987. Recall from that function $f: V(G) \to \{0,1\}$ is called Product Cordial if the induced function $f^*: E(G) \to \{0,1\}$ defined by $f^*(uv) = f(u)f(v)$ satisfies the condition $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for any $i, j \in \{0,1\}$, where $v_f(x)$ and $e_f(x)$ denotes the number of vertices and number of edges with label x respectively. A Graph G is called Product Cordial graph if its admit Product Cordial labeling. Let $G = (V(G), E(G))$ be a simple, undirected graph and $\{v_1, v_2, \cdots v_n\} \subseteq V(G)$, we call $v_1, v_2, \cdots v_n$ are in sequence if it forms a path. It is useful to recall some useful definitions of graph theory to make this article self-contained.

Definition 1. [1] A simple graph G is said to be complete if every pair of distinct vertices of G are adjacent in G. It is denoted by K_n.
Definition 2. [1] A walk in a graph G is a finite alternating sequence of vertices and edges. A walk is called a trail if all the edges are distinct. Cycle is a closed trail in which all the vertices are distinct. It is denoted by C_n.
Definition 3. [1] Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. Then union of G_1 and G_2 is denoted by $G_1 \cup G_2$ is the graphs whose vertex set is $V_1 \cup V_2$ and egde set is $E_1 \cup E_2$. When G_1 and G_2 are vertex disjoint $G_1 \cup G_2$ is called sum of G_1 and G_2 and it is denoted by $G_1 + G_2$.
Definition 4. [1] Let G_1 and G_2 be two vertex disjoint graphs. Then the join of G_1 and G_2 denoted by $G_1 \vee G_2$ is the supergraph of $G_1 + G_2$ in which each vertex of G_1 is also adjacent to every vertex of G_2.
Definition 5. [1] A graph is bipartite if is its vertex set can be partitioned into two non-empty subsets V_1 and V_2 such that each edge of G has one end in V_1 and other in V_2. The pair (V_1, V_2) is called bipartition of a bipartite graph. It is denoted by $G(V_1, V_2)$. A simple bipartite graph $G(V_1, V_2)$ is complete if each vertex of V_1 is adjacent to all the vertices of V_2. If $G(V_1, V_2)$ is complete with $|V_1| = m$ and $|V_2| = n$ then $G(V_1, V_2)$ is denoted by $K_{m,n}$.

In the next section, we have discussed some interesting results on Product Cordial Labeling.

2   Main Results

Theorem 2.1. Complete bipartite graph $K_{m,n}$ is not Product Cordial, where $m,n \geq 2$.
Proof. Without loss of generality, we may assume $m \geq n$. Let $V(K_{m,n}) = V_1 \cup V_2$. Where, $|V_1| = n$ and $|V_2| = m$. Suppose that $K_{m,n}$ is Product Cordial.
Case 1: $n + m$ is even
Since $K_{m,n}$ is Product Cordial, we have $v_f(0) = \frac{n+m}{2} = v_f(1)$. Suppose that there exist t vertices with label 0 in V_1. So, we have $n - t$ vertices with label 1 in V_1. Hence, there exists $\left(\frac{m+n}{2} - t\right)$ vertices with label 0 in V_2 and $m - \left(\frac{m+n}{2} - t\right) = \left(\frac{m-n}{2} + t\right)$ vertices with label 1 in V_2. Note that, $e_f(0) = mt + (n-t)\left(\frac{m+n}{2} - t\right)$ and $e_f(1) = (n-t)\left(m - \frac{m+n}{2} + t\right)$. Now, $e_f(0) - e_f(1) = mt + (n-t)\left(\frac{m+n}{2} - t\right) - (n-t)\left(m - \frac{m+n}{2} + t\right) = mt + (n-t)^2 - nt + t^2 > 2$.
Case 2: $n + m$ is odd
In this case we have two possibilities
(i) $v_f(0) = \frac{n+m+1}{2}$ and $v_f(1) = \frac{n+m-1}{2}$
(ii) $v_f(0) = \frac{n+m-1}{2}$ and $v_f(1) = \frac{n+m+1}{2}$
So, we consider the following cases.
Subase 2.1: Suppose that $v_f(0) = \frac{n+m+1}{2}$ and $v_f(1) = \frac{n+m-1}{2}$
Suppose that there exist t vertices with label 0 in V_1. So, we have, $(n-t)$ vertices with label 1 in V_1. Hence, there exists $\left(\frac{m+n+1}{2} - t\right)$ vertices with label 0 in V_2 and $\left[m - \left(\frac{m+n+1}{2} - t\right)\right] = \left(\frac{m-n-1}{2} + t\right)$ vertices with label 1 in V_2. Note that, $e_f(0) = mt + (n-t)\left(\frac{m+n+1}{2} - t\right)$ and $e_f(1) = (n-t)\left(\frac{m-n-1}{2} + t\right)$. Now, $e_f(0) - e_f(1) = mt + (n-t)\left(\frac{m+n+1}{2} - t\right) - (n-t)\left(\frac{m-n-1}{2} + t\right) = mt + (n-t)^2 + n - nt + t^2 - t > 2$.
Subase 2.2: Suppose that $v_f(0) = \frac{n+m-1}{2}$ and $v_f(1) = \frac{n+m+1}{2}$
Suppose that there exist t vertices with label 0 in V_1. So, we have, $(n-t)$ vertices with label 1 in V_1. Hence, there exists $\left(\frac{m+n-1}{2} - t\right)$ vertices with label 0 in V_2 and $\left[m - \left(\frac{m+n-1}{2} - t\right)\right] = \left(\frac{m-n+1}{2} + t\right)$ vertices with label 1 in V_2. Note that, $e_f(0) = mt + (n-t)\left(\frac{m+n-1}{2} - t\right)$ and $e_f(1) = (n-t)\left(\frac{m-n+1}{2} + t\right)$. Now, $e_f(0) - e_f(1) = mt + (n-t)\left(\frac{m+n+1}{2} - t\right) - (n-t)\left(\frac{m-n+1}{2} + t\right) = mt + (n-t)^2 - n + nt > 2$. Hence, $K_{m,n}$ is not Product Cordial.

Proposition 2.2. $K_n \vee C_n$ is not Product Cordial, where $n \geq 2$.
Proof. Suppose that $K_n \vee C_n$ is Product Cordial. Note that, $|V(K_n \vee C_n)| = 2n$ and $|E(K_n \vee C_n)| = n\frac{(n-1)}{2} + n + n^2$. Since, $|V(K_n \vee C_n)| = 2n$ and we have assumed that $K_n \vee C_n$ is Product Cordial.
We have $v_f(0) = v_f(1) = n$.
Case 1: *Suppose that all the vertices of label 0 and label 1 are in sequence in C_n*
Suppose that we have t no. of vertices with label 0 in K_n. So, we have $(n-t)$ vertices of of label 0 in C_n. Hence, we have $(n-t)$ vertices of label 1 in K_n and t vertices of label 1 in C_n. Note that, $e_f(0) = (n-t)t + t\frac{(t-1)}{2} + (n-t)^2 + (n-t+1) + nt$ and $e_f(1) = \frac{(n-t)(n-t-1)}{2} + t(n-t) + (t-1)$. Now, $e_f(0) - e_f(1) = \frac{n^2}{2} + t^2 + \frac{3n}{2} - 3t + 2$. If $t \geq 3$ then as $n \geq 2$, we have $e_f(0) - e_f(1) > 1$.
If $t = 1$ then $e_f(0) - e_f(1) = \frac{n^2}{2} + \frac{3n}{2} > 1$.
If $t = 2$ then $e_f(0) - e_f(1) = \frac{n^2}{2} + \frac{3n}{2} > 1$.
So, $e_f(0) - e_f(1) > 1$.
Case 2: *Suppose that some of the vertices of label 1 are not in sequence in C_n*
Suppose that we have t no. of vertices with label 0 in K_n. So, we have $(n-t)$ vertices of label 0 in C_n. Hence, we have $(n-t)$ vertices of label 1 in K_n and t vertices of label 1 in C_n. Suppose that there exist i no. of vertices with label 1 are not in sequence in C_n. Then, we have $e_f(0) = \frac{t(t-1)}{2} + t(n-t) + tn + (n-t)^2 + (n-t+i+1)$ and $e_f(1) = (t-i-1) + \frac{(n-t-1)(n-t)}{2} + t(n-t)$ Now, $e_f(1)$ in case 2 $\leq e_f(1)$ in case 1 and $e_f(0)$ in case 2 $\geq e_f(0)$ in case 1. So, $e_f(0) - e_f(1)$ in this case $\geq e_f(0) - e_f(1)$ in case 1. Now, we have already proved in case 1 that $e_f(0) - e_f(1) > 1$. Hence, in this case $e_f(0) - e_f(1) > 1$.
Case 3: *Suppose that we have n no. of vertices with label 0 in K_n and n no. of vertices with label 1 in C_n*
Then, we have $e_f(0) = \frac{n(n-1)}{2} + n^2$ and $e_f(1) = n$. Then, $e_f(0) - e_f(1) = \frac{n(n-1)}{2} + n^2 - n = \frac{3n^2}{2} - \frac{3n}{2} > 1$ as $n^2 > n$.
Case 4: *Suppose that we have n no. of vertices with label 1 in K_n and n no. of vertices with label 0 in C_n*

Then we have, $e_f(0) = n^2 + n$ and $e_f(1) = \frac{n(n-1)}{2}$. Then, $e_f(0) - e_f(1) = n^2 + n - \frac{n(n-1)}{2} = \frac{n^2}{2} + \frac{3n}{2} > 1$. Hence, $K_n \vee C_n$ is not Product Cordial.

Proposition 2.3. $K_n \vee C_m$ is not Product Cordial, where m + n is even and m,n ≥ 2.

Proof. Note that, $|V(K_n \vee C_m)| = m + n$. Suppose that $K_n \vee C_m$ is Product Cordial. Then we have, $|v_f(0)| = \frac{m+n}{2} = |v_f(1)|$.

Case 1: *Suppose that all the vertices with label 0 and label 1 are in sequence in C_m*

Suppose that we have t no. of vertices with label 0 in K_n. So, we have $\left(\frac{m+n}{2} - t\right)$ vertices with label 0 in C_m.

Hence, we have $(n - t)$ vertices with label 1 in K_n and $m - \left(\frac{m+n}{2} - t\right) = \left(\frac{m-n}{2} + t\right)$ vertices with label 1 in C_m.

Then we have, $e_f(0) = \frac{t(t-1)}{2} + tm + (n - t)\left(\frac{m+n}{2} - t\right) + \left(\frac{m+n}{2} - t + 1\right) + t(n - t)$ and $e_f(1) = \frac{(n-t)(n-t-1)}{2} + (n - t)\left(\frac{m-n}{2} + t\right) + \left(\frac{m-n}{2} + t - 1\right)$. Then, $e_f(0) - e_f(1) = mt + \frac{n^2}{2} - nt + \frac{3n}{2} + t^2 - 3t + 2 = (t - n)^2 \left(\frac{1}{2}\right) + \frac{t^2}{2} + \frac{3n}{2} + 2 + t(m - 3)$. If $m \geq 3$, then $e_f(0) - e_f(1) > 1$.

If $m = 2$, then $e_f(0) - e_f(1) = (t - n)^2 \left(\frac{1}{2}\right) + \frac{t^2}{2} + \frac{n}{2} + (n - t) + 2$.

Now, $n > t$. So, $e_f(0) - e_f(1) > 1$.

Case 2: *Suppose that some of the vertices with label 1 are not in sequence in C_m*

Suppose that we have t no. of vertices with label 0 in K_n. So, we have $\left(\frac{m+n}{2} - t\right)$ vertices with label 0 in C_m.

Hence, we have $(n - t)$ vertices with label 1 in K_n and $m - \left(\frac{m+n}{2} - t\right) = \left(\frac{m-n}{2} + t\right)$ vertices with label 1 in C_m.

Suppose that there exist i no. of vertices from $\left(\frac{m-n}{2} + t\right)$ with label 1 are not in sequence in C_m. Then we have, $e_f(0) = \frac{t(t-1)}{2} + t(n - t) + tm + (n - t)\left(\frac{m+n}{2} - t\right) + \left(\frac{m+n}{2} - t + i + 1\right)$ and $e_f(1) = \frac{(n-t)(n-t-1)}{2} + (n - t)\left(\frac{m-n}{2} + t\right) + \left(\frac{m-n}{2} + t - i - 1\right)$. Now, $e_f(1)$ in case 2 $\leq e_f(1)$ in case 1 and $e_f(0)$ in case 2 $\geq e_f(0)$ in case 1. So, $e_f(0) - e_f(1)$ in this case $\geq e_f(0) - e_f(1)$ in case 1. Now, we have already proved in case 1 that $e_f(0) - e_f(1) > 1$. Hence, $e_f(0) - e_f(1) > 1$ in this case.

Case 3: *Suppose that $m < n$*

Subcase 3.1: *Suppose that all the vertices in C_m are with label 0. Suppose that we have t no. of vertices with label 0 in K_n. So, we have $(n - t)$ vertices with label 1 in K_n. Then we have, $e_f(0) = \frac{t(t-1)}{2} + mn + m + t(n - t)$ and $\frac{(n-t)(n-t-1)}{2}$. Then, $e_f(0) - e_f(1) = mn + m + 2nt + \frac{n}{2} - t^2 - t - \frac{n^2}{2}$. We know that, $t = \frac{m+n}{2}$. Then, $e_f(0) - e_f(1) = \frac{3mn}{2} + \frac{m}{2} + \left(\frac{n^2}{4} - \frac{m^2}{4}\right)$. We know that $n > m$. So, $e_f(0) - e_f(1) > 1$.*

Subcase 3.2: *Suppose that all the vertices in C_m are with label 1. Suppose that we have t no. of vertices with label 0 in K_n. So, we have $(n - t)$ vertices with label 1 in K_n. Then we have, $e_f(0) = \frac{t(t-1)}{2} + tm + t(n - t)$ and $e_f(1) = \frac{(n-t)(n-t-1)}{2} + m(n - t) + m$. Then, $e_f(0) - e_f(1) = -m(n - t) + mt - m - \frac{n^2}{2} + 2nt + \frac{n}{2} - t^2 - t$. We know that, $t = \frac{m+n}{2}$. Then, $e_f(0) - e_f(1) = \frac{mn}{2} + \frac{3m^2}{4} + \frac{n^2}{4} - \frac{3m}{2}$. As $m \geq 2$. So, $e_f(0) - e_f(1) > 1$.*

Case 4: *Suppose that $m > n$*

Subcase 4.1: *Suppose that all the vertices in K_n are with label 0. Suppose that we have t no. of vertices with label 0 in C_m. So, we have $(m - t)$ vertices with label 1 in C_m.*

Subsubcase 4.1.1: *Suppose that all the vertices with label 1 are in sequence in C_m. Then we have, $e_f(0) = \frac{n(n-1)}{2} + (t + 1) + nm$ and $e_f(1) = m - t - 1$. Then, $e_f(0) - e_f(1) = \frac{n(n-1)}{2} + (t + 1) + nm - m + t + 1$. We know that, $nm > m$. So, $e_f(0) - e_f(1) > 1$.*

Subsubcase 4.1.2: *Suppose that some of the vertices with label 1 are not in sequence in C_m. Suppose that we have i no. of vertices with label 1 are not in sequence in C_m. Suppose that i no. of vertices are not i sequence. Then we have, $e_f(0) = \frac{n(n-1)}{2} + nm + (m + i + 1)$ and $e_f(1) = m - t - i - 1$. Now, $e_f(1)$ in subsubcase 4.1.2 $\leq e_f(1)$ in subsubcase 4.1.1 and $e_f(0)$ in subsubcase 4.1.2 $\geq e_f(0)$ in subsubcase 4.1.1. So, $e_f(0) - e_f(1)$ in this case $\geq e_f(0) - e_f(1)$ in subsubcase 4.1.1. Now, we have already proved in subsubcase 4.1.1 that $e_f(0) - e_f(1) > 1$. Hence, $e_f(0) - e_f(1) > 1$ in this case.*

Subcase 4.2: *Suppose that all the vertices in K_n are with label 1. Suppose that we have t no. of vertices with label 0 in C_m. So, we have $(m - t)$ vertices with label 1 in C_m.*

Subsubcase 4.2.1: *Suppose that all the vertices with label 1 are in sequence in C_m. Then we have, $e_f(0) = (t + 1) + nt$ and $e_f(1) = \frac{n(n-1)}{2} + (m - t - 1) + n(m - t)$. Then, $e_f(0) - e_f(1) = (t + 1) + tn - \frac{n(n-1)}{2} - m + t + 1 - mn + tn$. We know that, $t = \frac{m+n}{2}$. Then, $e_f(0) - e_f(1) = \frac{n^2}{2} + \frac{3n}{2} + 2 > 1$.*

Subsubcase 4.2.2: *Suppose that some of the vertices with label 1 are not in sequence in C_m. Suppose that i no. of vertices are not i sequence. Then we have, $e_f(0) = tn$ and $e_f(0) = \frac{n(n-1)}{2} + (m-t-i-1) + n(m-t)$. Now, $e_f(1)$ in subsubcase 4.2.2 $\leq e_f(1)$ in subsubcase 4.2.1 and $e_f(0)$ in subsubcase 4.2.2 $\geq e_f(0)$ in subsubcase 4.2.1. So, $e_f(0) - e_f(1)$ in this case is $\geq e_f(0) - e_f(1)$ in subsubcase 4.2.1. Now, we have already proved in subsubcase 4.2.1 that $e_f(0) - e_f(1) > 1$. Hence, $e_f(0) - e_f(1) > 1$ in this case. Hence, $K_n \vee C_m$ is not Product Cordial, where $m+n$ is even and $m,n \geq 2$.*

Proposition 2.4. $K_n \vee C_m$ is not Product Cordial, where m + n is odd and m,n ≥ 2.
Proof. Note that, $|V(K_n \vee C_m)| = m + n$. Suppose that $K_n \vee C_m$ is Product Cordial.
Case 1: *Suppose that all the vertices with label 0 and label 1 in C_m are in sequence in C_m*
In this case we have two possibilities
(i) $v_f(0) = \frac{m+n+1}{2}$ and $v_f(1) = \frac{m+n-1}{2}$
(ii) $v_f(0) = \frac{m+n-1}{2}$ and $v_f(1) = \frac{m+n+1}{2}$
So, we consider the following cases.
Subcase 1.1: *Suppose that $v_f(0) = \frac{m+n+1}{2}$ and $v_f(1) = \frac{m+n-1}{2}$*

Suppose that we have t no. of vertices with label 0 in K_n. So, we have $\left(\frac{m+n+1}{2} - t\right)$ vertices of label 0 in C_m. Hence, we have $(n-t)$ vertices with label 1 in K_n and $m - \left(\frac{m+n+1}{2} - t\right) = \left(\frac{m-n-1}{2} + t\right)$ vertices with label 1 in C_m. Then we have, $e_f(0) = \frac{t(t-1)}{2} + tn + t(n-t) + \frac{m+n+1}{2} - t + 1 + (n-t)\left(\frac{m+n+1}{2} - t\right)$ and $e_f(1) = \frac{(n-t)(n-t-1)}{2} + (n-t)\left(\frac{m-n-1}{2} + t\right) + \frac{m-n-1}{2} + t - 1$. Then, $e_f(0) - e_f(1) = \frac{n^2}{2} + \frac{5n}{2} + t^2 - 4t + 3 = (t-1)^2 + \frac{n^2}{2} + 2 + \frac{n}{2} + 2(n-t) > 1$ as $n > t$.
Subcase 1.2: *Suppose that $v_f(0) = \frac{m+n-1}{2}$ and $v_f(1) = \frac{m+n+1}{2}$*

Suppose that we have t no. of vertices with label 0 in K_n. So, we have $\left(\frac{m+n-1}{2} - t\right)$ vertices of label 0 in C_m. Hence, we have $(n-t)$ vertices with label 1 in K_n and $m - \left(\frac{m+n-1}{2} - t\right) = \left(\frac{m-n+1}{2} + t\right)$ vertices with label 1 in C_m. Then we have, $e_f(0) = \frac{t(t-1)}{2} + tm + t(n-t) + \frac{m+n-1}{2} - t + 1 + (n-t)\left(\frac{m+n-1}{2} - t\right)$ and $e_f(1) = \frac{(n-t)(n-t-1)}{2} + (n-t)\left(\frac{m-n+1}{2} + t\right) + \frac{m-n+1}{2} + t - 1$. Then, $e_f(0) - e_f(1) = t^2 + \frac{n^2}{2} + \frac{n}{2} + mt + 1 - 2t - nt = \frac{(n-1)^2}{2} + (t-1)^2 + mt + \frac{n}{2} - \frac{1}{2} > 1$ as $n \geq 2$.
Case 2: *Suppose that some of the vertices with label 1 are not in sequence in C_m*
Subcase 2.1: *Suppose that $v_f(0) = \frac{m+n+1}{2}$ and $v_f(1) = \frac{m+n-1}{2}$*

Suppose that we have t no. of vertices with label 0 in K_n. So, we have $\left(\frac{m+n+1}{2} - t\right)$ vertices of label 0 in C_m. Hence, we have $(n-t)$ vertices with label 1 in K_n and $m - \left(\frac{m+n+1}{2} - t\right) = \left(\frac{m-n-1}{2} + t\right)$ vertices with label 1 in C_m. Suppose that there exist i no. of vertices from $\left(\frac{m-n-1}{2} + t\right)$ with label 1 are not in sequence in C_m. Then we have, $e_f(0) = \frac{t(t-1)}{2} + tm + t(n-t) + \left(\frac{m+n+1}{2} - t + i + 1\right) + (n-t)\left(\frac{m+n+1}{2} - t\right)$ and $e_f(1) = \left(\frac{m-n-t}{2} - t - i - 1\right) + \frac{(n-t)(n-t-1)}{2} + (n-t)\left(\frac{m-n-1}{2} + t\right)$. Now, $e_f(1)$ in subcase 2.1 $\leq e_f(1)$ in subcase 1.1 and $e_f(0)$ subcase 2.1 $\geq e_f(0)$ in subcase 1.1. So, $e_f(0) - e_f(1)$ in this case $\geq e_f(0) - e_f(1)$ in subcase 1.1. Now, we have already proved in subcase 1.1 that $e_f(0) - e_f(1) > 1$. Hence, $e_f(0) - e_f(1) > 1$ in this case.
Subcase 2.2: *Suppose that $v_f(0) = \frac{m+n-1}{2}$ and $v_f(1) = \frac{m+n+1}{2}$*

Suppose that we have t no. of vertices with label 0 in K_n. So, we have $\left(\frac{m+n-1}{2} - t\right)$ vertices of label 0 in C_m. Hence, we have $(n-t)$ vertices with label 1 in K_n and $m - \left(\frac{m+n-1}{2} - t\right) = \left(\frac{m-n+1}{2} + t\right)$ vertices with label 1 in C_m. Suppose that there exist i no. of vertices from $\left(\frac{m-n+1}{2} + t\right)$ with label 1 are not in sequence in C_m. Then we have, $e_f(0) = \frac{t(t-1)}{2} + tm + t(n-t) + \left(\frac{m+n-1}{2} - t + i + 1\right) + (n-t)\left(\frac{m+n-1}{2} - t\right)$ and $e_f(1) = \left(\frac{m-n+1}{2} + t - i - 1\right) + \frac{(n-t)(n-t-1)}{2} + (n-t)\left(\frac{m-n+1}{2} + t\right)$. Now, $e_f(1)$ in subcase 2.2 $\leq e_f(1)$ in subcase 1.2. and $e_f(0)$ subcase 2.2 $\geq e_f(0)$ in subcase 1.2. So, $e_f(0) - e_f(1)$ in this case $\geq e_f(0) - e_f(1)$ in subcase 2.1. Now, we have already proved in subcase 2.1 that $e_f(0) - e_f(1) > 1$. Hence, $e_f(0) - e_f(1) > 1$ in this case.
Case 3: *Suppose that $m < n$*
Subcase 3.1: *Suppose that all the vertices in C_m are with label 0 and some vertices with label 0 are in K_n. Suppose that there exist t no. of vertices with label 0 in K_n. So, there exists $(n-t)$ vertices with label 1 in K_n. Suppose that we have m no. of vertices with label 0 in C_m. Then we have, $e_f(0) = \frac{t(t-1)}{2} + t(n-t) + mn + m$*

and $e_f(1) = \frac{(n-t)(n-t-1)}{2}$. Then, $e_f(0) - e_f(1) = mn + m + 2nt + \frac{n}{2} - t - t^2 - \frac{n^2}{2}$.

In this case we have two possibilities

(i) $m + t = \frac{m+n+1}{2}$

(ii) $m + t = \frac{m+n-1}{2}$

So, we consider the following cases.

Subsubcase 3.1.1: *Suppose that $m + t = \frac{m+n+1}{2}$. Therefore, $t = \frac{n-m+1}{2}$. Then, $e_f(0) - e_f(1) = \frac{mn}{2} + \left(2m - \frac{3}{4}\right) + \left(\frac{n^2}{4} - \frac{m^2}{4}\right) + \frac{n}{2} > 1$ as $m < n$ and $2m > \frac{3}{4}$ as $m \geq 2$.*

Subsubcase 3.1.2: *Suppose that $m + t = \frac{m+n-1}{2}$. Thenrefore, $t = \frac{n-m-1}{2}$. Then, $e_f(0) - e_f(1) = \left(\frac{mn}{2} - \frac{n}{2}\right) + m + \left(\frac{n^2}{4} - \frac{m^2}{4}\right) + \frac{1}{4} > 1$ as $n > m$.*

Subcase 3.2: *Suppose that all the vertices in C_m are with label 1 and some vertices with label 1 are in K_n. Suppose that there exist t no. of vertices with label 0 in K_n. So, there exists $(n - t)$ vertices with label 1 in K_n. Suppose that we have m no. of vertices with label 1 in C_m. Then we have, $e_f(0) = \frac{t(t-1)}{2} + t(n - t) + tm$ and $e_f(1) = \frac{(n-t)(n-t-1)}{2} + m(n - t) + m$. Then, $e_f(0) - e_f(1) = 2mt - mn - \frac{n^2}{2} + 2nt + \frac{n}{2} - t^2 - t - m$.*

Subsubcase 3.2.1: *Suppose that $t = \frac{m+n+1}{2}$. Then, $e_f(0) - e_f(1) = \frac{3m^2}{4} + \left(\frac{mn}{2} - m\right) + \left(\frac{n^2}{4} - \frac{3}{4}\right) + \frac{n}{2} > 1$ as $m, n \geq 2$.*

Subsubcase 3.2.2: *Suppose that $t = \frac{m+n-1}{2}$. Then, $e_f(0) - e_f(1) = \frac{3m^2}{4} + \frac{mn}{2} - 2m + \frac{n^2}{4} + \frac{1}{4} - \frac{n}{2} = \left(\frac{n^2}{4} - \frac{n}{2}\right) + m\left(\frac{3m}{4} + \frac{n}{2} - 2\right) + \frac{1}{4} > 1$ as $m, n \geq 2$.*

Case 4: *Suppose that $m > n$ and Suppose that all the vertices with label 1 are in sequence in C_m*

Subcase 4.1: *Suppose that all the vertices in K_n are with label 0 and some vertices with label 0 are in C_m. Suppose that there exist t no. of vertices with label 0 in C_m. So, there exists $(m - t)$ vertices with label 1 in C_m. Suppose that we have n no. of vertices with label 0 in K_n. Then we have, $e_f(0) = mn + (t + 1) + n\frac{(n-1)}{2}$ and $e_f(1) = m - t - 1$. Then, $e_f(0) - e_f(1) = (mn - m) + 2 + 2t + \left(\frac{n^2}{2} - \frac{n}{2}\right) > 1$ as $mn > m$ and $\frac{n^2}{2} > \frac{n}{2}$, where, $m, n \geq 2$.*

Subcase 4.2: *Suppose that all the vertices in K_n are with label 1 and some vertices with label 1 are in C_m. Suppose that there exist t no. of vertices with label 0 in C_m. So, there exists $(m - t)$ vertices with label 1 in C_m. Suppose that we have n no. of vertices with label 1 in K_n. Then we have, $e_f(0) = tn + (t + 1)$ and $e_f(1) = \frac{n(n-1)}{2} + n(m - t) + (m - t - 1)$. Then, $e_f(0) - e_f(1) = 2t + 2 + 2nt - \frac{n^2}{2} + \frac{n}{2} - mn - m$.*

Subsubcase 4.2.1: *Suppose that $t = \frac{m+n+1}{2}$. Then, $e_f(0) - e_f(1) = \frac{5n}{2} + \frac{n^2}{2} + 3 > 1$.*

Subsubcase 4.2.2: *Suppose that $t = \frac{m+n-1}{2}$. Then, $e_f(0) - e_f(1) = \frac{n^2}{2} + \frac{n}{2} + 1 > 1$.*

Case 5: *Suppose that $m > n$ and Suppose that some of the vertices with label 1 are not in sequence in C_m*

Subcase 5.1: *Suppose that all the vertices in K_n are with label 0 and some vertices with label 0 are in C_m. Suppose that there exist t no. of vertices with label 0 in C_m. So, there exists $(m - t)$ vertices with label 1 in C_m. Suppose that we have n no. of vertices with label 0 in K_n. Suppose that we have i no. of vertices with label 1 are not in sequence in C_m. Then, $e_f(0) = \frac{n(n-1)}{2} + mn + (t + i + 1)$ and $e_f(1) = m - t - i - 1$. Now, $e_f(1)$ in subcase 5.1 $\leq e_f(1)$ in subcase 4.1 and $e_f(0)$ in subsubcase 5.1 $\geq e_f(0)$ in subsubcase 4.1. So, $e_f(0) - e_f(1)$ in this case is $\geq e_f(0) - e_f(1)$ in subcase 4.1. Now, we have already proved in subcase 4.1 that $e_f(0) - e_f(1) > 1$. Hence, $e_f(0) - e_f(1) > 1$ in this case.*

Subcase 5.2: *Suppose that all the vertices in K_n are with label 1 and some vertices with label 1 are in C_m. Suppose that there exist t no. of vertices with label 0 in C_m. So, there exists $(m - t)$ vertices with label 1 in C_m. Suppose that we have n no. of vertices with label 1 in K_n. Suppose that we have i no. of vertices with label 1 are not in sequence in C_m. Then $e_f(0) = nt + (t + i + 1)$ and $e_f(1) = \frac{n(n-1)}{2} + (m - t - i - 1)$. Now, $e_f(1)$ in subcase 5.2 $\leq e_f(1)$ in subcase 4.2 and $e_f(0)$ in subsubcase 5.2 $\geq e_f(0)$ in subsubcase 4.2. So, $e_f(0) - e_f(1)$ in this case is $\geq e_f(0) - e_f(1)$ in subcase 4.2. Now, we have alredy proved in subcase 4.2 that $e_f(0) - e_f(1) > 1$. Hence, $e_f(0) - e_f(1) > 1$ in this case. Hence, $K_n \vee C_m$ is not Product Cordial, where $m + n$ is odd and $m, n \geq 2$.*

Theorem 2.5. $K_n \vee C_m$ is not Product Cordial, where m,n ≥ 2, m,n ∈ **N**.

Proof. Proof follows from Propositions 2.2, 2.3 and 2.4.

Proposition 2.6. $C_m \vee C_n$ is not Product Cordial, where m = n and m ≥ 2.

Proof. Suppose that $C_m \vee C_n$ is Product Cordial for m = n. Note that, $|V(C_m \vee C_n)| = 2n$ and $|E(C_m + C_n)| = n + m + nm = 2n + n^2$ as n = m. Since, $|V(C_m \vee C_n)| = m + n = 2n$ as n = m . We have assumed that $C_m \vee C_n$ is Product Cordial for n = m. We have $v_f(0) = v_f(1) = n$.

Case 1: *Suppose that all the vertices of label 0 are in sequence in C_m and C_n*

Then, it is clear that all the vertices of label 1 are in sequence in C_m and C_n. Suppose that we have t no. of vertices with label 0 in C_m. So, we have $(n - t)$ vertices of of label 0 in C_n. Hence, we have $(m - t)$ vertices of label 1 in C_m and t vertices of label 1 in C_n. Note that, $e_f(0) = (t + 1) + (n - t + 1) + tn + (n - t)n$ and $e_f(1) = (n - t - 1) + (t - 1) + t(n - t)$. Then, $e_f(0) - e_f(1) = 2n + 2 + tn + n^2 - t^2$. We know that, $n > t$. So, $e_f(0) - e_f(1) > 1$.

Case 2: *Suppose that some of the vertices of label 1 are not in sequence in C_m and C_n*

Suppose that we have t no. of vertices with label 0 in C_m. So, we have $(n - t)$ vertices of label 0 in C_n. Hence, we have $(m - t)$ vertices of label 1 in C_m and t vertices of label 1 in C_n. Suppose that there exist i no. of vertices with label 1 are not in sequence in C_m and j no. of vertices with label 1 are not in sequence in C_n. Note that, $e_f(0) = (t + i + 1) + (n - t + j + 1) + tn + (n - t)m$ and $e_f(1) = (n - t - i - 1) + (t - j - 1) + t(n - t)$. Now, $e_f(1)$ in case 2 $\leq e_f(1)$ in case 1 and $e_f(0)$ in case 2 $\geq e_f(0)$ in case 1. So, $e_f(0) - e_f(1)$ in this case is $\geq e_f(0) - e_f(1)$ We have already proved in case 1 that $e_f(0) - e_f(1) > 1$. Hence, $e_f(0) - e_f(1) > 1$ in this case.

Case 3: *Suppose that we have m no. of vertices with label 0 in C_m and n no. of vertices with label 1 in C_n*

Note that, $e_f(0) = mn + m$ and $e_f(1) = n$. Then, $e_f(0) - e_f(1) = mn + m - n = mn > 1$ as n = m. ***Case 4:*** *Suppose that we have m no. of vertices with label 1 in C_m and n no. of vertices with label 0 in C_n*

Note that, $e_f(0) = mn + n$ and $e_f(1) = m$. Then, $e_f(0) - e_f(1) = mn + n - m > 1$ as n = m. Hence, $C_m \vee C_n$ is not Product Cordial, where m = n and m ≥ 2.

Proposition 2.7. $C_m \vee C_n$ is not Product Cordial, where m + n is even and m,n ≥ 2.

Proof. Note that, $|V(C_n \vee C_m)| = n + m$. Suppose that $C_n \vee C_m$ is Product Cordial. Since we have $v_f(0) = \frac{n+m}{2} = v_f(1)$.

Case 1: *Suppose that all the vertices of label 0 and 2 are in sequence in C_n and C_m*

Suppose that we have t no. of vertices with label 0 in C_n. So, we have $(n - t)$ vertices of of label 1 in C_n. Hence, we have $\frac{n+m}{2} - t$ vertices of label 0 in C_m and $m - \frac{n+m}{2} + t$ vertices of label 1 in C_m. Note that, $e_f(0) = (t + 1) + \left(\frac{n+m}{2} - t + 1\right) + tm + \left(\frac{n+m}{2} - t\right)(n - t)$ and $e_f(1) = (n - t - 1) + \left(m - \frac{n+m}{2} + t - 1\right) + (n - t)\left(m - \frac{n+m}{2} + t\right)$. Then, $e_f(0) - e_f(1) = mt + 4 + n^2 - 3nt + 2t^2$. We know that $t = \frac{m+n}{2}$, So, we have $e_f(0) - e_f(1) = m^2 + 4 > 1$.

Case 2: *Suppose that some of the vertices of label 1 are not in sequence in C_n and C_m*

Suppose that we have t no. of vertices with label 0 in C_n. So, we have $\frac{n+m}{2} - t$ vertices of label 0 in C_m.

Hence, we have $(n - t)$ vertices of label 1 in C_n and $\left(m - \frac{n+m}{2} + t\right)$ vertices of label 1 in C_m. Suppose that there exist i no. of vertices with label 1 are not in sequence in C_n and j no. of vertices with label 1 are not in sequence in C_m. Note that, $e_f(0) = (t + i + 1) + \left(\frac{n+m}{2} - t + j + 1\right) + tm + (n - t)\left(\frac{n+m}{2} - t\right)$ and $e_f(1) = (n - t - i - 1) + \left(m - \frac{n+m}{2} + t - j - 1\right) + (n - t)\left(m - \frac{n+m}{2} + t\right)$. Now, $e_f(1)$ in case 2 $\leq e_f(1)$ in case 1 and $e_f(0)$ in case 2 $\geq e_f(0)$ in case 1. So, $e_f(0) - e_f(1)$ in this case is $\geq e_f(0) - e_f(1)$ in case 1. Now, we have already proved in case 1 that $e_f(0) - e_f(1) > 1$. Hence, in this case $e_f(0) - e_f(1) > 1$.

Case 3: *Suppose that $m > n$*

Subcase 3.1: *Suppose that all the vertices in C_n are with label 0. So, we have n no. of vertices with label 0 in C_n. Suppose that we have t no. of vertices with label 0 in C_m. So, there exist $m - t$ no. of vertices with label 1 in C_m.*

Subsubcase 3.1.1: *Suppose that all the vertices in c_m are in sequence. Then, $e_f(0) = n + (t + 1) + mn + tn$ and $e_f(1) = m - t - 1$. Then, $e_f(0) - e_f(1) = (mn - m) + n + 2t + tn + 2 > 1$ as mn > m.*

Subsubcase 3.1.2: *Suppose that all the vertices with label 1 are not in sequence in c_m. Suppose that we have i no. of vertices from $(m - t)$ no. of vertices are not in sequence in c_m. Then, $e_f(0) = n + (t + i + 1) + mn$ and $e_f(1) = m - t - i - 1$. Now, $e_f(1)$ in subsubcase 3.1.2 $\leq e_f(1)$ in subsubcase 3.1.1 and $e_f(0)$ in subsubcase 3.1.2 $\geq e_f(0)$ in subsubcase 3.1.1. So, $e_f(0) - e_f(1)$ in this case $\geq e_f(0) - e_f(1)$ in subsubcase 3.1.1. Now, we have already proved in subsubcase 3.1.1 that $e_f(0) - e_f(1) > 1$. Hence, $e_f(0) - e_f(1) > 1$ in this case.*

Subcase 3.2: *Suppose that all the vertices in C_n are with label 1. So, we have n no. of vertices with label 1 in C_n. Suppose that we have t no. of vertices with label 0 in C_m. So, there exist $m - t$ no. of vertices with label*

1 in C_m.

Subsubcase 3.2.1: *Suppose that all the vertices in c_n are in sequence. Then, $e_f(0) = t + 1 + tn$ and $e_f(1) = n + m - t - 1$. Then, $e_f(0) - e_f(1) = nt - n - m + 2t + 2$. We know that $t = \frac{n+m}{2}$. So, $e_f(0) - e_f(1) = \frac{mn}{2} + \frac{n^2}{2} + 2 > 1$.*

Subsubcase 3.2.2: *Suppose that all the vertices in c_n are not in sequence. Suppose that we have i no. of vertices from $(n - t)$ no. of vertices are not in sequence in c_m. Then, $e_f(0) = t + i + 1 + tn$ and $e_f(1) = m - t - i - 1 + n + n(m - t)$. Now, $e_f(1)$ in subsubcase 3.2.2 $\leq e_f(1)$ in subsubcase 3.2.1 and $e_f(0)$ in subsubcase 3.2.2 $\geq e_f(0)$ in subsubcase 3.2.1. so, $e_f(0) - e_f(1)$ in this case $\geq e_f(0) - e_f(1)$ in subsubcase 3.2.1. Now, we have already proved in subsubcase 3.2.1 that $e_f(0) - e_f(1) > 1$. Hence, $e_f(0) - e_f(1) > 1$ in this case. Hence, $C_m \vee C_n$ is not Product Cordial, where $n + m$ is even and $m, n \geq 2$.*

Proposition 2.8. $C_m \vee C_n$ is not Product Cordial, where m + n is odd and m,n ≥ 2.
Proof. Note that, $|V(C_n \vee C_m)| = n + m = 2k + 1$. Suppose that $C_n \vee C_m$ is Product Cordial. Without loss of generality we may assume that $m > n$.

In this case we have two possibilities. $(i) v_f(0) = \frac{m+n+1}{2}$ and $v_f(1) = \frac{m+n-1}{2}$

$(ii) v_f(0) = \frac{m+n-1}{2}$ and $v_f(1) = \frac{m+n+1}{2}$

So, we consider the following cases.

Case 1: *$v_f(0) = \frac{n+m+1}{2} = k + 1$ and $v_f(1) = \frac{n+m-1}{2} = k$*

Subcase 1.1: *Suppose that all the vertices of label 0 are in sequence in C_n and C_m*
Then, it is clear that all the vertices of label 1 are in sequence in C_n and C_m. Suppose that we have t no. of vertices with label 0 in C_n. So, we have $(n - t)$ vertices of of label 1 in C_n. Hence, we have $(k + 1 - t)$ vertices of label 0 in C_m and $(k - n + t)$ vertices of label 1 in C_m. Note that, $e_f(0) = (t + 1) + (k + 2 - t) + tm + (k + 1 - t)(n - t)$ and $e_f(1) = (n - t - 1) + (k - n + t - 1) + (n - t)(k - n + t)$. Then, $e_f(0) - e_f(1) = (n - t)^2 + 5 + tm + (n - t)(1 - t) = (n - t)(n + 1 - 2t) + tm + 5$. Now, $e_f(0) - e_f(1) > 1$ if $n + 1 \geq 2t$. If $n + 1 < 2t$, then $\frac{(n+1)}{2} < t$. Now, $t + k = \frac{m+n+1}{2} > \frac{(n+t)}{2} + k$. Therefore, $m > k$. Suppose that $t = \frac{(n+1)}{2} + l$. Then, $e_f(0) - e_f(1) = 2l^2 + 2l + \frac{1}{2} + lm + 5 > 1$.

Subcase 1.2: *Suppose that some of the vertices of label 1 are not in sequence in C_n and C_m*
Suppose that we have t no. of vertices with label 0 in C_n. So, we have $(n - t)$ vertices of label 1 in C_n. Hence, we have $(k - t)$ vertices of label 0 in C_m and $(k + 1 - n + t)$ vertices of label 1 in C_m. Suppose that there exist i no. of vertices with label 1 are not in sequence in C_n and j no. of vertices with label 1 are not in sequence in C_m. Note that, $e_f(0) = (t + i + 1) + (k - t + j + 2) + tm + (n - t)(k + 1 - t)$ and $e_f(1) = (n - t - i - 1) + (k - n + t - j - 1) + (n - t)(k - n + t)$. Now, $e_f(1)$ in subcase 1.2 $\leq e_f(1)$ in subcase 1.1 and $e_f(0)$ in subcase 1.2 $\geq e_f(0)$ in subcase 1.1. So, $e_f(0) - e_f(1)$ in this case is $\geq e_f(0) - e_f(1)$ in subcase 1.1. Now, we have already proved in subcase 1.1 that $e_f(0) - e_f(1) > 1$. Hence, $e_f(0) - e_f(1) > 1$ in this case .

Case 2: *$v_f(0) = \frac{n+m-1}{2} = k$ and $v_f(1) = \frac{n+m+1}{2} = k + 1$*

Subcase 2.1: *Suppose that all the vertices of label 0 are in sequence in C_n and C_m*
Then, it is clear that all the vertices of label 1 are in sequence in C_n and C_m. Suppose that we have t no. of vertices with label 0 in C_n. So, we have $(n - t)$ vertices of label 1 in C_n. Hence, we have $(k + 1 - t)$ vertices of label 0 in C_m and $(k - n + t)$ vertices of label 1 in C_m. Note that, $e_f(0) = (t + 1) + (k + 1 - t) + tm + (k - t)(n - t)$ and $e_f(1) = (n - t - 1) + (k - n + t) + (n - t)(k - n + t + 1)$. Then, $e_f(0) - e_f(1) = (n - t)^2 + 3 + tm + (t - n)(1 + t) = (n - t)(n - 1 - 2t) + tm + 3$. Now, $e_f(0) - e_f(1) > 1$ if $n \geq 1 + 2t$. If $n < 1 + 2t$, then $\frac{(n-1)}{2} < t$. Now, $t + k = \frac{m+n-1}{2} > \frac{(n-t)}{2} + k$. Therefore, $m > k$. Suppose that $t = \frac{(n-1)}{2} + l$. Then, $e_f(0) - e_f(1) = \left(\frac{mn}{2} - \frac{m}{2}\right) + 3 + l(m - n - 1) > 1$, if $m \geq n + 1$. Suppose that $m \leq n + 1$. Then since, $m \geq n$, we have $m = n + 1$. So, we have $e_f(0) - e_f(1) = \left(\frac{mn}{2} - \frac{m}{2}\right) + 3 + l(m - n - 1) = \left(\frac{mn}{2} - \frac{m}{2}\right) + 3 > 1$.

Subcase 2.2: *Suppose that some of the vertices of label 1 are not in sequence in C_n and C_m*
Suppose that we have t no. of vertices with label 0 in C_n. So, we have $(n - t)$ vertices of label 1 in C_n. Hence, we have $(k - t)$ vertices of label 0 in C_m and $(k + 1 - n + t)$ vertices of label 1 in C_m. Suppose that there exist i no. of vertices with label 1 are not in sequence in C_n and j no. of vertices with label 1 are not in sequence in C_m. Note that, $e_f(0) = (t + i + 1) + (k - t + j + 1) + tm + (n - t)(k - t)$ and $e_f(1) = (n - t - i - 1) + (k - n + t - j) + (n - t)(k - n + t + 1)$. Now, $e_f(1)$ in subcase 2.2 $\leq e_f(1)$ in subcase 2.1 and $e_f(0)$ in subcase 2.2 $\geq e_f(0)$ in subcase 2.1. So, $e_f(0) - e_f(1)$ in this case is $\geq e_f(0) - e_f(1)$ in subcase 2.1. Now, we have already proved in subcase 2.1 that $e_f(0) - e_f(1) > 1$. Hence, $e_f(0) - e_f(1) > 1$ in this case.

Case 3: *Suppose that $m > n$*
Subcase 3.1: *Suppose that all the vertices in C_n are with label 0. So, we have n no. of vertices with label 0 in C_n. Suppose that we have t no. of vertices with label 0 in C_m. So, there exist $m - t$ no. of vertices with label*

1 in C_m.

Subsubcase 3.1.1: *Suppose that all the vertices in C_m are in sequence. Then, $e_f(0) = n + (t + 1) + mn + tn$ and $e_f(1) = m - t - 1$. Then, $e_f(0) - e_f(1) = (mn - m) + n + 2t + tn + 2 > 1$ as $mn > m$.*

Subsubcase 3.1.2: *Suppose that all the vertices with label 1 are not in sequence in C_m. Suppose that we have i no. of vertices from $(m - t)$ no. of vertices are not in sequence in C_m. Then, $e_f(0) = n + (t + i + 1) + mn$ and $e_f(1) = m - t - i - 1$. Now, $e_f(1)$ in subsubcase 3.1.2 $\leq e_f(1)$ in subsubcase 3.1.1 and $e_f(0)$ in subsubcase 3.1.2 $\geq e_f(0)$ in subsubcase 3.1.1. So, $e_f(0) - e_f(1)$ in this case is $\geq e_f(0) - e_f(1)$ in subsubcase 3.1.1. Now, we have already proved in subsubcase 3.1.1 that $e_f(0) - e_f(1) > 1$. Hence, $e_f(0) - e_f(1) > 1$ in this case.*

Subcase 3.2: *Suppose that all the vertices in C_n are with label 1. So, we have n no. of vertices with label 1 in C_n. Suppose that we have t no. of vertices with label 0 in C_m. So, there exist $m - t$ no. of vertices with label 1 in C_m.*

Subsubcase 3.2.1: *Suppose that all the vertices in C_n are in sequence. Then, $e_f(0) = t + 1 + tn$ and $e_f(1) = n + m - t - 1$. Then, $e_f(0) - e_f(1) = nt - n - m + 2t + 2$.*
In this case we have two possibilities.

(i) Suppose that $t = \frac{n+m+1}{2}$. So, $e_f(0) - e_f(1) = \frac{mn}{2} + \frac{n^2}{2} + \frac{n}{2} + 3 > 1$

(ii) Suppose that $t = \frac{n+m-1}{2}$. So, $e_f(0) - e_f(1) = \frac{mn}{2} + \left(\frac{n^2}{2} - \frac{n}{2}\right) + 3 > 1$.

Subsubcase 3.2.2: *Suppose that all the vertices in C_n are not in sequence. Suppose that we have i no. of vertices from $(n - t)$ no. of vertices are not in sequence in C_m. Then, $e_f(0) = t + i + 1 + tn$ and $e_f(1) = m - t - i - 1 + n + n(m - t)$. Now, $e_f(1)$ in subsubcase 3.2.2 $\leq e_f(1)$ in subsubcase 3.2.1 and $e_f(0)$ in subsubcase 3.2.2 $\geq e_f(0)$ in subsubcase 3.2.1. So, $e_f(0) - e_f(1)$ in this case is $\geq e_f(0) - e_f(1)$ in subsubcase 3.2.1. Now, we have already proved in subsubcase 3.2.1 that $e_f(0) - e_f(1) > 1$. Hence, $e_f(0) - e_f(1) > 1$ in this case. Hence, $C_m \vee C_n$ is not Product Cordial, where $n + m$ is odd and $m, n \geq 2$.*

Theorem 2.9. $C_m \vee C_n$ is not Product Cordial, where n,m $\in$ **N**, m,n $\geq$ 2.
Proof. Proof follows from propositions 2.6, 2.7 and 2.8.

3   Conclusion

In this article we have proved that Complete bipartite graphs $K_{m,n}$, $K_n \vee C_m$ and $C_n \vee C_m$ is not Product Cordial graphs.

References

1. R. Balakrishnan and K. Ranganathan, A Textbok of Graph Theory, New York:Springer-Verlag, 2012.
2. Cahit, Cordial graphs : A weaker version of Graceful and Harmonious Graphs, Ars Combinatorial, 23(3), 1987, 201-207.
3. K. Chitra Lakshmi, K. Nagarjan, Geometric Mean Cordial Labeling of Graphs, International Journal of Mathematics and Soft Computing, 7(1), (2017), 75-87.
4. J. A. Gallian, A dynamic survey of graph labeling, The Electronic Journal of Combinatorics, 19, #DS6, 2022.
5. J. Gross and J. Yellen, Graph Theory and its Applications (Second edition), Boca Raton : CRC press, 2006.
6. F. Harary, Graph Theory, Addison wisely, New Delhi, 1969.
7. R. Ponraj and M. Sivakumar, M. Sundaram, Mean Cordial Labeling of Graphs, Open Journal of Discrete Mathematics, 2(4), 145-148.
8. M. Sundaram, R. Ponraj and S. Somasundaram, Product cordial labeling of graphs, Bull. Pure and Appl. Sci. (Math. & Stat.), 23E (2004) 155-163.
9. S. Mundadiya, J. Parejiya and M. Jariya, Root Cube Mean Cordial Labeling of $C_n \vee C_m$, for n,m $\in$ N, TWMS J. App. and Eng. Math., 14(2), (2024), 460-472
10. S. Mundadiya, J. Parejiya, H. Gandhi, R. Solanki and M. Jariya, Root Cube Mean Cordial Labeling of $K_{1,n} \times P_m$, AIP Conference Proceedings, 2963(1), (2023), 020020-22.