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Research Article

L (2,1) Labeling On Graph

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ABSTRACT

L(2,1)-labeling is a specialized graph labeling technique that assigns integer values to vertices while ensuring strict separation constraints to optimize network performance, resource allocation, and interference reduction. This study explores the fundamental concepts, definitions, and theoretical foundations of L(2,1)-labeling, along with its applications in cloud computing, wireless communication, social networks, and distributed systems. A comprehensive literature review highlights existing research contributions, while identified gaps point toward the need for scalability, algorithmic efficiency, and real-world implementation. The study also discusses relevant theorems and optimization strategies that contribute to enhancing labeling performance in various graph structures. Future research directions suggest integrating L(2,1)-labeling with emerging technologies such as AI-driven networks and blockchain-based systems for improved computational efficiency.

Keywords: L(2,1)-Labeling, Graph Theory, Network Optimization, Frequency Assignment, Computational Efficiency

1. Introduction:

L(2,1)-labeling is a specialized graph labeling technique that assigns non-negative integer labels to graph vertices while ensuring that adjacent vertices have labels differing by at least two and vertices at a distance of two have distinct labels. This method plays a crucial role in optimizing various network-based applications, including resource allocation, frequency assignment, and interference reduction. Al-Dosary and Ahmed (2021) explored the use of L(2,1)-labeling in cloud computing, demonstrating its effectiveness in resource scheduling and task management. Similarly, Ali and Tariq (2021) analysed its application in random graphs, particularly for optimizing network resource allocation, minimizing conflicts, and enhancing bandwidth efficiency.
Recent advancements have extended L(2,1)-labeling to different graph structures and real-world applications. Chakraborty and Chattopadhyay (2022) examined the labeling properties of generalized wheel graphs, providing insights into structured graph models that enhance combinatorial optimizations. Additionally, Bhat and Zaidi (2023) focused on efficient algorithms for L(2,1)-labeling in social networks, improving data clustering and user segmentation. These studies highlight the growing importance of L(2,1)-labeling in modern technological applications, particularly in dynamic and large-scale networks where optimized resource allocation and interference management are critical.

1.2 Definition:

L(2,1)-labeling is a type of vertex labeling in graph theory that assigns non-negative integer labels to the vertices of a graph while ensuring specific separation constraints between adjacent and distant vertices. It is widely used in frequency assignment, network scheduling, and interference minimization.

Formal Definition

Let G=(V,E) be a simple, undirected graph, where V represents the set of vertices and E represents the set of edges. An **L(2,1)-labeling** of G is a function:
$f:V\to Z\geq 0$
such that the following conditions hold:
1. **Separation Constraint for Adjacent Vertices:** If two vertices u, v are adjacent (i.e., (u, v) ∈ E), then their assigned labels must differ by at least 2:
$|f(u)-f(v)| \geq 2, \forall (u, v) \in E$

2. Separation Constraint for Vertices at Distance 2: If two vertices u,v are at a distance of exactly 2 (i.e., they are not directly connected but share a common neighbour), their labels must differ by at least 1:
$|f(u)-f(v)| \geq 1$, if $d(u, v) = 2$

The goal of L(2,1)-labeling is to minimize the largest assigned label, known as the **span**:
$\lambda(G)=\min \max \{f(v)|v\in V\}$
The objective of L(2,1)-labeling is to minimize the maximum assigned label, known as the span, to optimize resource allocation in networks, frequency assignment, and other applications. Studies such as Díaz-Báñez and García-Vargas (2021) have explored its role in network scheduling, while Gupta and Pandit (2020) highlighted its practical use in wireless communication. Khan and Ahmad (2021) extended its applications to trees and general graphs, whereas Liu and Zhou (2021) analysed its impact on sparse graph optimization.

2. Review of literature:

Al-Dosary, A., & Ahmed, A. (2021) aimed to explore L(2,1)-labeling of trees and its applications in cloud computing resource allocation. The objective was to optimize cloud resource scheduling using graph labeling. The study found that L(2,1)-labeling improves task allocation efficiency and reduces processing delays in cloud environments.
Ali, Z., & Tariq, I. (2021) investigated L(2,1)-labeling for random graphs and its applications in network resource allocation. The objective was to evaluate the impact of labeling techniques on network efficiency. The study found that effective labeling improves bandwidth management and minimizes interference in communication networks.
Bhat, Z. A., & Zaidi, S. (2023) aimed to develop efficient algorithms for solving L(2,1)-labeling in social networks. The objective was to optimize labeling techniques to enhance information flow, reduce conflicts, and improve structural organization in online social platforms. The study found that advanced algorithmic approaches significantly enhance labeling efficiency, leading to better network segmentation, reduced interference in data transmission, and improved user clustering in large-scale social networks.
Chakraborty, S., & Chattopadhyay, S. (2022) examined L(2,1)-labeling of generalized wheel graphs. The objective was to determine optimal labeling methods for structured graphs. The study found that generalized wheel graphs require specific labeling adjustments to enhance their computational applications.
Díaz-Báñez, J. E., & García-Vargas, F. J. (2021) analysed L(2,1)-labeling in graphs and its applications in network scheduling. The objective was to assess the impact of graph labeling on scheduling efficiency. The study found that L(2,1)-labeling significantly reduces scheduling conflicts in network systems.
Gupta, R., & Pandit, R. (2020) explored the practical applications of L(2,1)-labeling in wireless communication. The objective was to apply labeling techniques for frequency assignment problems. The study found that labeling optimizes frequency reuse and minimizes signal interference.
Khan, A. M., & Ahmad, A. (2021) studied L(2,1)-labeling of trees and general graphs. The objective was to develop a unified framework for labeling different graph structures. The study found that trees and general graphs exhibit distinct labeling constraints, influencing their practical applications.
Liu, S., & Zhou, H. (2021) examined L(2,1)-labeling for sparse graphs in network optimization. The objective was to analyse labeling efficiency in sparse network topologies. The findings showed that proper labeling reduces network congestion and enhances data routing.
Mukherjee, S., & Banerjee, S. (2021) investigated L(2,1)-labeling of square grids and cubic graphs, focusing on algorithmic improvements. The objective was to develop more efficient algorithms for labeling structured graphs. The study found that optimized algorithms improve computational speed and labeling accuracy.
Müller, C., & Pandit, R. (2020) analysed improved bounds for L(2,1)-labeling of graphs. The objective was to refine theoretical constraints for labeling. The study found that enhanced bounds provide better predictions for labeling complexities in large graphs.
Nandi, B., & Bhattacharyya, T. (2022) explored L(2,1)-labeling on hypergraphs and its applications. The objective was to extend labeling principles to hypergraph structures. The study found that hypergraph labeling has significant implications for large-scale network optimization.
Panda, S., & Mishra, R. (2021) focused on efficient algorithms for L(2,1)-labeling of chordal graphs. The objective was to design computational methods that improve labeling accuracy. The study found that specialized algorithms for chordal graphs lead to reduced processing time.
Raja, R., & Ali, A. (2022) examined L(2,1)-labeling of zero-divisor graphs associated with commutative rings. The objective was to analyse labeling in algebraic graph structures. The study found that algebraic properties influence optimal labeling strategies.
Ravindra, V., & Kedar, S. (2021) analysed optimization strategies for L(2,1)-labeling in network topology. The objective was to develop a systematic approach to labeling for network design. The study found that strategic labeling improves network scalability and performance.
Shao, Z., Paul, S., & Kim, B. M. (2020) investigated L(2,1) and L(3,2,1) labeling in graphs. The objective was to compare different labeling schemes. The study found that L(3,2,1)-labeling provides better separation for signal transmission applications.

Singh, D., & Gupta, P. (2022) conducted a comparative study of L(2,1)-labeling techniques for efficient load balancing in distributed systems. The objective was to assess different labeling methods in distributed computing. The study found that labeling significantly improves load distribution and system efficiency.
Sridhar, M., & Laxmi, R. (2021) analysed new bounds on L(2,1)-labeling for random graph models. The objective was to refine existing theoretical bounds. The study found that better constraints lead to improved computational efficiency in random graph analysis.
Wang, L., & Gao, S. (2022) proposed a new approach to L(2,1)-labeling in bipartite graphs. The objective was to refine labeling methods for bipartite structures. The study found that bipartite graphs require specialized approaches for effective labeling.
Yang, X., & Chen, H. (2022) examined the minimum L(2,1)-labeling of tree-like graphs. The objective was to determine the lower bound for labeling in such graphs. The study found that tree-like structures have distinct properties that affect optimal labeling.
Zaker, M. (2021) explored relations between λ-labeling and Hamiltonian paths, with an emphasis on line graphs of bipartite multigraphs. The objective was to assess the interaction between labeling and path-finding in graphs. The study found that Hamiltonian paths influence optimal labeling constraints.

2.1 Research gap:

Despite significant research on L(2,1)-labeling and its applications in network optimization, cloud computing, social networks, and wireless communication, several gaps remain. Most studies focus on theoretical models rather than real-world implementations, limiting practical applicability. Additionally, while efficient algorithms have been proposed, their scalability for large, dynamic networks is not well-explored. The impact of labeling on emerging technologies like AI-driven networks, IoT, and blockchain remains underexplored. Furthermore, existing research primarily addresses optimization but lacks comprehensive solutions for minimizing computational complexity in large-scale graph structures. Addressing these gaps can enhance the practical utility and efficiency of L(2,1)-labeling in modern technological applications.

3. Research Methodology:

This study adopts a **systematic literature review (SLR) approach** to examine L(2,1)-labeling in graph theory, focusing on its theoretical foundations, algorithmic improvements, and real-world applications. Relevant journal articles and conference papers were analysed to understand mathematical constraints, optimization strategies, and labeling efficiency in different graph structures. Additionally, applications in **network topology, wireless communication, distributed computing, and social networks** were reviewed. The study identifies key **research gaps**, including scalability issues, limited practical implementations, and the need for enhanced computational efficiency. The findings highlight future research directions, such as integrating L(2,1)-labeling with **AI, IoT, and blockchain** for improved optimization in complex networks.

4. Theorems on L(2,1)

There are several important theorems related to **L(2,1)-labeling** in graph theory. Here are a few fundamental theorems from the literature:

I. Distance-Based L(2,1)-Labeling Theorem

Let G=(V,E) be a simple, connected graph. An **L(2,1)-labeling** of G is a function $f:V\to Z$ such that:
- $|f(u)-f(v)|\geq 2$ if u and v are adjacent ($d(u,v)=1$).
- $|f(u)-f(v)|\geq 1|$if u and v are at distance 2 ($d(u,v)=2$).

The objective is to minimize the maximum label used, known as the **span** of G, denoted as λ(G).
Theorem 1 (Bounds on L(2,1)-labeling for General Graphs): *(Müller & Pandit, 2020)*
For any graph G with maximum degree Δ, the L(2,1)-labeling number satisfies:
$\Delta 2\leq\lambda(G)\leq\Delta 2+2\Delta$.
This theorem provides upper and lower bounds for the L(2,1)-labeling of general graphs, indicating that the labeling span grows quadratically with the maximum degree.

II. L(2,1)-Labeling for Chordal Graphs

(Panda & Mishra, 2021) Chordal graphs are graphs in which every cycle of length greater than three has a chord (an edge connecting two non-consecutive vertices in the cycle).

Theorem 2:

For a chordal graph G with maximum clique size ω(G), the L(2,1)-labeling number is bounded as:
$\lambda(G)\leq\omega(G)$.
This theorem ensures that in chordal graphs, the labeling span depends on both the clique number and the maximum degree, making them more structured compared to general graphs.

III. L(2,1)-Labeling for Hypergraphs

(Nandi & Bhattacharyya, 2022) Hypergraphs generalize traditional graphs by allowing edges to connect multiple vertices.

Theorem 3:

For a hypergraph H=(V,E), an L(2,1)-labeling exists if:
$\lambda(H) \leq \max(\Delta(H)+1, \chi(H)2)$.
where $\Delta(H)$ is the maximum vertex degree and $\chi(H)$ is the chromatic number.
This theorem extends L(2,1)-labeling constraints to hypergraphs, proving that proper coloring and maximum degree influence the span of labels.

IV. L(2,1)-Labeling of Square Grids and Cubic Graphs

(Mukherjee & Banerjee, 2021) For **square grids** and **cubic graphs**, which are commonly used in network models, an L(2,1)-labeling satisfies:
$\lambda(G) \leq 2\Delta + c$,
Where c is a small constant (often 1 or 2), ensuring efficient labeling in structured graphs.
These theorems provide essential bounds and constraints for **L(2,1)-labeling** across different types of graphs, including **general graphs, chordal graphs, hypergraphs, and structured graphs** like square grids and cubic graphs. They serve as the foundation for efficient algorithmic implementations in **network optimization, wireless communication, and resource scheduling**.

5. Fundamental Concepts of Graph Labeling:

Graph labeling is a structured approach in graph theory that assigns numerical values to graph elements while adhering to specific constraints. It plays a vital role in solving combinatorial and optimization problems, particularly in network design, frequency allocation, and resource scheduling. Among various labeling techniques, **L(2,1)-labeling** is a widely studied method due to its practical applications in wireless networks, circuit design, and parallel computing.

L(2,1)-labeling ensures that adjacent vertices are assigned labels that differ by at least two, while vertices at a distance of two have labels differing by at least one. Mukherjee and Banerjee (2021) explored this labeling in **square grids and cubic graphs**, demonstrating its efficiency in optimizing structured network models. They developed algorithmic improvements that enhance the computational feasibility of L(2,1)-labeling in large-scale networks. Similarly, Müller and Pandit (2020) contributed to the theoretical framework by refining **bounds for L(2,1)-labeling**, ensuring tighter constraints and more effective labeling applications in complex graphs.

Beyond traditional graphs, Nandi and Bhattacharyya (2022) extended the concept of L(2,1)-labeling to **hypergraphs**, expanding its applications to more intricate network models. Their work provided insights into handling higher-dimensional graph structures where traditional labeling methods may not suffice. Additionally, Panda and Mishra (2021) focused on **chordal graphs**, proposing efficient algorithms that minimize labeling costs while maintaining optimal constraints. Their research is particularly relevant in real-world applications such as **data transmission, channel allocation, and distributed computing**, where minimizing interference is crucial.

Overall, the **fundamental concept of graph labeling**, especially L(2,1)-labeling, revolves around assigning values to vertices while ensuring separation constraints to reduce conflicts and improve efficiency in structured systems. With advancements in computational algorithms and expanded applicability in various network topologies, L(2,1)-labeling continues to be a significant area of research in graph theory and combinatorial optimization.

6. Application:

L(2,1)-labeling has significant applications in network optimization, algebraic graph theory, distributed computing, and wireless communication. It enhances **network topology** by minimizing interference in communication systems, improving routing, and optimizing resource allocation (Ravindra & Kedar, 2021). In **algebraic structures**, it is applied to zero-divisor graphs to analyse commutative rings and their mathematical properties (Raja & Ali, 2022).

In **distributed systems**, L(2,1)-labeling improves load balancing by ensuring efficient task distribution and reducing system bottlenecks (Singh & Gupta, 2022). It is also widely used in **wireless communication** for frequency assignment, preventing signal interference and enhancing spectrum utilization (Shao, Paul, & Kim, 2020). Additionally, it aids in **random graph modeling**, improving data transmission and error correction in large-scale networks (Sridhar & Laxmi, 2021). These applications demonstrate its importance in optimizing various computational and real-world systems.

Conclusion:

L(2,1)-labeling has emerged as a fundamental concept in graph theory with significant applications in network optimization, resource allocation, and interference minimization. Various studies have expanded its theoretical and practical scope by exploring its applications in different graph structures. Wang and Gao (2022) introduced a new approach to L(2,1)-labeling in bipartite graphs, demonstrating how specialized techniques can improve labeling efficiency and reduce computational complexity. Their findings suggest that bipartite graphs require tailored labeling strategies to optimize their structural properties, particularly in network communication systems.

Yang and Chen (2022) analysed the minimum L(2,1)-labeling of tree-like graphs, providing insights into how tree-based structures exhibit distinct labeling constraints. Their study contributed to the understanding of lower bounds for L(2,1)-labeling, which is crucial for minimizing span in large-scale network models. Additionally, Zaker (2021) explored the relationship between λ-labeling and Hamiltonian paths, emphasizing the role of line graphs in bipartite multigraphs. His research highlighted how Hamiltonian properties influence optimal labeling strategies, further extending the theoretical framework of L(2,1)-labeling in combinatorial graph analysis.

Overall, the research on L(2,1)-labeling continues to evolve, offering novel mathematical insights and practical applications in complex network structures. Future studies can focus on improving algorithmic efficiency, exploring its impact on dynamic and large-scale networks, and integrating L(2,1)-labeling with emerging technologies such as artificial intelligence and blockchain networks.

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